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# Analytical and Numerical Investigation of Mixed-Type Functional Differential Equations

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## Abstract

This paper is concerned with the approximate solution of a linear non-autonomous functional differential equation, with both advanced and delayed arguments. We search for a solution  $x(t)$ , defined for  $t \in [-1, k], (k \in \mathbb{N})$ , that satisfies this equation almost everywhere on  $[0, k - 1]$  and assumes specified values on the intervals  $[-1, 0]$  and  $(k - 1, k]$ . We provide a discussion of existence and uniqueness theory for the problems under consideration and describe numerical algorithms for their solution, giving an analysis of their convergence.

*Key words:* Mixed-type functional differential equation, method of steps, collocation method, splines, least squares method

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## 1. Introduction

This paper is concerned with mixed-type functional differential equations (MTFDEs). To be precise, we shall consider equations of the type

$$x'(t) = \alpha(t)x(t) + \beta(t)x(t - 1) + \gamma(t)x(t + 1), \quad (1)$$

where  $x$  is the unknown function,  $\alpha$ ,  $\beta$  and  $\gamma$  are known functions. Interest in this type of equation is motivated by applications in optimal control [17], economic dynamics [18], nerve conduction [2] and travelling waves in a spatial lattice [1]. Important contributions to their analysis have appeared in the literature in the two last decades of the past century. The ill-posedness of MTFDEs was discussed by Rustichini in 1989 [17], where he considered linear autonomous equations. The same author extended his results to nonlinear equations [18]. J. Mallet-Paret applied Fredholm theory to obtain new results for this class of equation [13] and introduced the idea of factorization of their solutions [14]. Independently, the authors of [7] obtained results about factorization for the linear non-autonomous case. Additionally, Krisztin [11] analysed the roots of the characteristic equation of linear systems of MTFDE, which has led to important results on the qualitative behaviour of their solutions. In particular, he has shown that a MTFDE may have a nonoscillatory solution in spite of the non-existence of real roots of its characteristic equation (by contrast with delay-differential equations). More recently, H.J. Hupkes and S. Verduyn Lunel studied the behaviour of solutions to nonlinear autonomous MTFDEs in the neighborhood of an equilibrium solution [8]. They have shown that the solutions that remain sufficiently close to an equilibrium can be captured on a

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finite dimensional invariant center manifold. This theory has been successfully applied to the analysis of an economic life-cycle model. In [9], the same authors generalised their results to the case of a periodic solution (instead of an equilibrium one). In [10], a class of mixed-type difference equations has been studied, which has the property that their solutions automatically satisfy an associated MTFDE. Using this connection, a finite dimensional center manifold has been determined also for this class of difference equations.

Based on the existing insights on the qualitative behaviour of MTFDEs, the authors of [4] and [5] recently developed a new approach to the analysis of these equations in the autonomous case. More precisely, they analysed MTFDEs as boundary value problems, that is, for a MTFDE of the form (1), but with constant coefficients  $\alpha$ ,  $\beta$  and  $\gamma$ , they considered the problem of finding a differentiable solution on a certain real interval  $[-1, k]$ ,  $k \in \mathbb{N}$ , given its values on the intervals  $[-1, 0]$  and  $(k-1, k]$ . They concluded that in general the specification of such boundary functions is not sufficient to ensure that a solution can be found. For the case where such a solution exists they introduced a numerical algorithm to compute it. This approach was further developed in [19]<sup>1</sup>, where new numerical methods were proposed for the solution of such boundary value problems. In [20], these methods were extended to the non-autonomous case (when  $\alpha$ ,  $\beta$  and  $\gamma$  are smooth functions of  $t$ ). Though interesting numerical results were obtained in both papers, many important questions remained open, in particular, a complete numerical analysis of the methods used was not provided. In the present paper, our main goal is to give a solid theoretical basis to the computational methods, by relating them to the existing analytical results about MTFDEs and to classical results of numerical analysis.

In this paper we are concerned only with equations of the form (1), that is, linear MTFDEs. However, many problems arising in applications, like those considered in [1] and [2], are nonlinear. Thus it is not possible at this stage to compare our methods with those presented in [1] and [2]. We expect that in the near future our approach can be developed further and applied to nonlinear problems with real world applications.

The outline of this article is as follows. In Section 2, we consider the question of existence of solutions. We present the “method of steps” for the non-autonomous equation (1), a technique used to extend a certain solution of this equation, given on the interval  $[-1, 1]$ , to a larger interval. We use this idea to extend our understanding of the behaviour of solutions as well as giving sufficient conditions for a solution of a particular class to exist. In Section 3, we recall existing insights about uniqueness of solutions for the class of problems considered in this work. We are able to give a new result on uniqueness for the equations under consideration here. In Section 4, we revisit the computational methods introduced in [19] and [20], and analyse their order of convergence. Finally, in Section 5, we present numerical results that illustrate the theoretical results for the numerical methods. The paper concludes with some remarks on open questions and further work.

## 2. Fundamental properties, existence theory and the method of steps

Our ultimate goal is to compute a particular solution of equation (1) which satisfies the boundary conditions

$$x(t) = \begin{cases} \varphi_1(t), & \text{if } t \in [-1, 0], \\ f(t), & \text{if } t \in (k-1, k], \end{cases} \quad (2)$$

where  $\varphi_1$  and  $f$  are smooth real-valued functions, defined on  $[-1, 0]$  and  $(k-1, k]$ , respectively ( $1 < k \in \mathbb{N}$ ).

We begin by discussing properties of a solution. We require that the equation (1) is satisfied for almost all  $t \in (0, k-1]$  (actually, we require that (1) is satisfied except possibly at the integer values of  $t$ ). To avoid pathological cases (which we shall mention later), we also assume that our solution is continuous on  $[-1, k]$  and has bounded variation. It follows that  $x'(t)$  is continuous wherever (1) is satisfied on  $(0, k-1)$ . On  $(1, k-2)$  one can differentiate (1) and conclude that  $x''(t)$  is continuous wherever (1) is satisfied on  $(1, k-2)$  and the process can be repeated. We can summarise by saying that the solution may have a discontinuity in the first derivative at  $t = 0$  and/or  $t = k-1$  and becomes progressively smoother on this sequence of internal sub-intervals.

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<sup>1</sup>this article is also available electronically at <http://www.springerlink.com/content/n580366744501411/fulltext.pdf>

In order to analyse and solve this boundary value problem (BVP) of (1) subject to (2) we consider first an initial value problem (IVP), with the conditions:

$$x(t) = \varphi(t), \quad t \in [-1, 1], \quad (3)$$

where the function  $\varphi$  is defined by

$$\varphi(t) = \begin{cases} \varphi_1(t), & \text{if } t \in [-1, 0], \\ \varphi_2(t), & \text{if } t \in (0, 1]. \end{cases} \quad (4)$$

This reformulation provides a basis for both analytical and numerical construction of solutions using ideas based on Bellman's method of steps for solving delay differential equations. One solves the equation over successive intervals of length unity. We need to assume the non-degeneracy condition that  $\gamma(t) \neq 0$ , for  $t \geq 0$ , so that equation (1) can be rewritten in the form

$$x(t+1) = a(t)x'(t) + b(t)x(t-1) + c(t)x(t), \quad t \geq 0 \quad (5)$$

where

$$a(t) = \frac{1}{\gamma(t)}, \quad b(t) = -\frac{\beta(t)}{\gamma(t)}, \quad c(t) = -\frac{\alpha(t)}{\gamma(t)}. \quad (6)$$

**Remark 2.1.** If  $x'$  is not defined for a particular value of  $t$ , then we shall use the value  $x'(t^-)$  in (5).

In principle, we can use formula (5) to construct a solution of equation (1) on an interval  $(1, k]$ , starting from its definition on  $[-1, 1]$  using the starting values given by (4).

So, for example, if  $a, b, c \in C^1([0, 3])$ , and supposing that all the appropriate derivatives of  $\varphi_i$  exist, we may obtain the following expressions for the solution in the first two intervals:

$$\begin{aligned} x(t) &= a(t-1)\varphi_2'(t-1) + b(t-1)\varphi_1(t-2) + c(t-1)\varphi_2(t-1), \quad t \in (1, 2]; \\ x(t) &= a(t-1)a(t-2)\varphi_2''(t-2) + [a(t-1)(a'(t-2) + c(t-2)) + c(t-1)a(t-2)]\varphi_2'(t-2) + \\ &+ [c'(t-2)a(t-1) + c(t-1)c(t-2) + b(t-1)]\varphi_2(t-2) + [a(t-1)b(t-2)]\varphi_1'(t-3) + \\ &+ [a(t-1)b'(t-2) + c(t-1)b(t-2)]\varphi_1(t-3), \quad t \in (2, 3]. \end{aligned} \quad (7)$$

We remark that these formulae reduce to the corresponding formulae of Table 1 in [4], if we set  $c(t) \equiv c$ ,  $a(t) \equiv a$ ,  $b(t) \equiv b$ .

Continuing this process, we can extend the solution to any interval, provided that the initial function  $\varphi$  and the functions  $a, b, c$  are smooth enough functions and satisfy some simple relationships. In a moment, we shall formulate this result in more precise terms but first we need to consider a little more closely the relationship between solutions of (1) subject to (2) and of (5) subject to (4).

As we already remarked, the solution of the BVP becomes smoother as we move away from the ends of the interval. However, the solution of the IVP, constructed using the method of steps, becomes *less smooth* as time increases. The conclusions on smoothness for the solution of the IVP constructed using the method of steps are summarised in the following Theorem:

**Theorem 2.1.** Let  $x$  be the solution of problem (5), (4), where  $\alpha(t), \beta(t), \gamma(t) \in C^{2L}([-1, 2L+1])$ ,  $\gamma(t) \neq 0$ ,  $t \in [-1, 2L+1]$ ,  $\varphi_1(t) \in C^{2L+1}([-1, 0])$ ,  $\varphi_2(t) \in C^{2L+1}([0, 1])$  (for some  $L \in \mathbb{N}$ ).

Moreover, suppose that

$$\varphi_1^{(\ell)}(0^-) = \varphi_2^{(\ell)}(0^+), \quad (8)$$

and

$$\begin{aligned} \varphi_2(1) &= a(0)\varphi_1'(0^-) + b(0)\varphi_1(-1) + c(0)\varphi_1(0); \\ \varphi_2^{(\ell)}(1^-) &= \frac{d^\ell}{dt^\ell} (a(t)\varphi_1'(t) + b(t)\varphi_1(t-1) + c(t)\varphi_1(t))|_{t=0^-}, \quad \ell = 0, 1, 2, \dots, 2L+1. \end{aligned} \quad (9)$$

Then there exist functions  $\delta_{i,l}, \epsilon_{i,l}, \bar{\delta}_{i,l}, \bar{\epsilon}_{i,l} \in C([-1, 2L+1])$ ,  $l = 1, \dots, L$ ,  $i = 0, 1, \dots, 2l$ , such that the following formulae are valid:

$$\begin{aligned} x(t) &= \sum_{i=0}^{2l-1} \delta_{i,l}(t) \varphi_1^{(i)}(t-2l) + \sum_{i=0}^{2l-1} \epsilon_{i,l}(t) \varphi_2^{(i)}(t-2l+1), & t \in [2l-1, 2l]; \\ x(t) &= \sum_{i=0}^{2l} \bar{\epsilon}_{i,l}(t) \varphi_2^{(i)}(t-2l) + \sum_{i=0}^{2l-1} \bar{\delta}_{i,l}(t) \varphi_1^{(i)}(t-2l-1), & t \in [2l, 2l+1]; \end{aligned} \quad l = 1, 2, \dots \quad (10)$$

Moreover, the solution  $x$ , constructed according to the formulae (10), belongs to the class

$$C^{2L+1}([-1, 1)) \cap C^{2L}([-1, 2)) \cap \dots \cap C^1([-1, 2L+1)).$$

In [4] analogous formulae were derived for the autonomous case. In the case of variable coefficients, formulae (10) can be obtained using similar arguments. A detailed proof can be found in the Technical Report [12].

**Remark 2.2.** If the assumptions of Theorem 2.1 are satisfied for some  $L \in \mathbb{N}$ , we can conclude that the solution  $x$ , constructed according to the formulae (10), has at least  $2L-l+1$  continuous derivatives on each interval  $[l, l+1)$ , for  $l \geq 1$ ; in other words, if the solution is given on  $[-1, 0]$  and  $(0, 1]$  by some functions  $\varphi_1, \varphi_2$ , belonging to some class  $C^{2L+1}([0, 1])$  and satisfying conditions (8), (9), its degree of smoothness decreases by exactly one on each subsequent subinterval.

Next we consider again the original BVP (1) subject to (2). If  $x'(r^-) \neq x'(r^+)$  (a jump in the derivative at  $t = r$ ) then that is almost certain to be reflected in a discontinuity in the solution at least at one of the points  $t = r, t = r-1, t = r+1$ . This means that the effect of derivative discontinuities in solutions of mixed type equations may be much more fundamental than those encountered in delay equations and, while they cannot be ignored, they are less likely to arise in practical applications where the solutions may be expected, at least, to be continuous. It is for this reason that, in the remainder of the paper, we shall not concern ourselves with problems where derivative and other discontinuities arise. It is sufficient to remark that these problems are inherently complex and cannot be readily dealt with using the methods from this paper. Inevitably such discontinuities in the solution also lead to a loss of order in numerical schemes (see [6]).

One important conclusion to draw from the preceding paragraph is that it is only very specific examples of mixed equations that have well-behaved solutions and this means that the underlying problem is itself highly unstable with respect to small perturbations in any of  $\alpha, \beta, \gamma, \varphi$ . It comes as no surprise that the method of steps is also highly unstable. As a result, numerical methods based on formula (5), even in the autonomous case, are very sensitive to small errors. This situation may become critical when the first derivative in this formula is approximated by finite differences. In this case, the numerical methods may fail even for moderate interval lengths. For example, in [4], where a numerical algorithm was used, based on formula (5) and on  $\theta$ -methods, it was verified that for  $k > 4$ , the method does not converge to the true solution, owing to instability. However, it turns out that these weaknesses of the method may be overcome by a careful decomposition of the solution into two components (a growing and a decaying one) and approximating these components separately. This approach makes use of the theoretical results on exponential dichotomies (see, for example, [7]) and has been analysed in a recently submitted paper [6].

In order to discuss the necessary conditions for the existence of at least one continuously differentiable solution to a given BVP of the form (1) subject to (2), we begin by describing the ODE approach, introduced in two recent papers [19], [20]. In the remainder of this section we will assume that the functions  $\alpha, \beta, \gamma, \varphi_1, f$  satisfy the conditions of theorem 2.1, so that the method of steps can be applied.

On the interval  $[-1, 1]$  the solution of (1), (2) can be written formally as

$$x(t) = x_0(t) + u(t), \quad t \in [-1, 1], \quad (11)$$

where  $x_0$  is some approximation of the solution and  $u$  is a correction. More precisely, we require that  $x_0$  satisfies the following conditions:

- a)  $x_0(t) = \varphi_1(t), \quad \forall t \in [-1, 0];$
- b)  $x_0$  is  $k$  times continuously differentiable on  $[-1, 0]$ ,  $x_0(0) = \varphi_1(0)$  and  $x_0^{(j)}(0) = \varphi_1^{(j)}(0^-)$ ,  $j = 1, \dots, k$ ;
- c)  $x_0$  is  $k-1$  times continuously differentiable on  $(0, 1]$ ,  $x_0(1) = x(1)$  and  $x_0^{(j)}(1^-) = x^{(j)}(1^-)$ ,  $j = 1, \dots, k-1$  (where  $x$  is the required solution of the problem (1),(2)).

Details about how to construct  $x_0$  will be given in the sequel.

Once  $x_0$  is defined, our problem is reduced to the computation of the correction  $u$ . First of all note that  $u(t) \equiv 0, \forall t \in [-1, 0]$  (otherwise,  $x$  does not satisfy the first boundary condition). Therefore, if we define  $u$  on  $[0, 1]$ , we can extend it to the whole interval  $[-1, k]$  using the method of steps (as described in Section 2). Let us denote by  $u^{[-1, k]}$  the extension of  $u$  to the interval  $[-1, k]$ . Then  $u^{[-1, k]}$  is defined, piece by piece, in the following way:

$$u^{[-1, k]}(t) = \begin{cases} 0, & \text{if } t \in [-1, 0]; \\ u(t), & \text{if } t \in (0, 1]; \\ u^{[l-1, l]}(t), & \text{if } t \in (l-1, l], \quad l = 2, 3, \dots, k. \end{cases} \quad (12)$$

We shall now express  $u^{[l-1, l]}(t)$  in terms of  $u(t)$ . From (1), taking into account that  $u(t) \equiv 0$ , for  $t \in [-1, 0]$ , we obtain

$$u^{[1, 2]}(t) = a(t-1)u'(t-1) + c(t-1)u(t-1), \quad t \in (1, 2], \quad (13)$$

where  $a$  and  $c$  are defined by (6). In the same way, we can define  $u^{[2, 3]}$ :

$$u^{[2, 3]}(t) = a(t-1)(u^{[1, 2]})'(t-1) + c(t-1)u^{[1, 2]}(t-1) + b(t-1)u^{[1, 2]}(t-1). \quad (14)$$

From (13) and (14) we conclude that

$$u^{[2, 3]}(t) = a(t-1)a(t-2)u''(t-2) + (a(t-1)a'(t-2) + a(t-1)c(t-2) + a(t-2)c(t-1))u'(t-2) + (c'(t-2)a(t-1) + c(t-1)c(t-2) + b(t-1)u(t-2)), \quad t \in (2, 3]. \quad (15)$$

Notice that formula (15) can be obtained from (7) considering  $\varphi_1(t) = 0$  and  $\varphi_2(t) = u(t)$ . Continuing this process, we can express  $u^{[l-1, l]}(t)$  in terms of  $u(t)$  and its derivatives, on any interval  $(l-1, l]$ , by an equation of the form

$$u^{[l-1, l]}(t) = c_{l-1, l}(t)u^{(l-1)}(t-l+1) + c_{l-2, l}(t)u^{(l-2)}(t-l+1) + \dots + c_{0, l}(t)u(t-l+1), \quad (16)$$

$t \in (l-1, l]$ ,  $l = 2, \dots, k$ . Here  $c_{il}$  are coefficients that can be computed recursively, just as the  $\delta_{il}$  and  $\epsilon_{il}$  coefficients in the right-hand side of (10). In particular, on the interval  $(k-1, k]$ , we obtain

$$u^{[k-1, k]}(t) = L_{k-1}u(t-k+1) := c_{k-1, k}(t)u^{(k-1)}(t-k+1) + \dots + c_{0, k}(t)u(t-k+1), \quad (17)$$

$t \in (k-1, k]$ .

Note that  $x_0$  can also be extended to the interval  $[-1, k]$ , using the same method. Let us denote by  $x_0^{[-1, k]}$  the extension of  $x_0$  to this interval. Then we conclude that  $x$  satisfies

$$x(t) = x_0^{[-1, k]}(t) + u^{[-1, k]}(t) = L_{k-1}u(t-k+1) + x_0^{[-1, k]}(t), \quad t \in [k-1, k]. \quad (18)$$

Now, since  $x$  must satisfy the second boundary condition in (2), we conclude that

$$L_{k-1}u(t-k+1) + x_0^{[-1, k]}(t) = f(t), \quad t \in [k-1, k], \quad (19)$$

or equivalently

$$L_{k-1}u(t) = f(t+k-1) - x_0^{[-1, k]}(t+k-1), \quad t \in [0, 1]. \quad (20)$$

Moreover, since  $u(t) = x(t) - x_0(t)$ ,  $\forall t \in [-1, 1]$ , and  $x_0$  satisfies the above described conditions, we conclude that  $u \in C^k([0, 1]) \cap C^{k-1}([0, 1])$  and the following boundary conditions must be satisfied:

$$\begin{aligned} u(0) &= u'(0) = \dots = u^{(k)}(0) = 0; \\ u(1) &= u'(1) = \dots = u^{(k-1)}(1) = 0. \end{aligned} \quad (21)$$

The number of boundary conditions in (21),  $2k + 1$ , is higher than the order of the considered ODE (20). Therefore, there may not exist a solution of (20) which satisfies all the conditions (21). This is not surprising, since the existence of a solution to the original boundary value problem (1), (2) is also not guaranteed (as discussed in [4]).

Hence, when solving the problem (20), (21) one has to keep only  $k - 1$  conditions and ignore the remaining ones. If  $k - 1$  is even, we consider  $(k - 1)/2$  boundary conditions at each end; if  $k - 1$  is odd, we consider  $k/2$  conditions at  $t = 0$  and  $k/2 - 1$  at  $t = 1$ . Let us call the obtained boundary value problem (with  $k - 1$  boundary conditions) the **reduced BVP**.

For example, in the case  $k = 3$ , (20) is a second order ODE and the reduced BVP has two boundary conditions:  $u(0) = 1, u(1) = 0$ . The obtained BVP can then be solved by standard numerical methods, for example, the collocation method with a basis of cubic B-splines. For details, see Section 4.

Let us now analyse under which conditions on the coefficients of the problem (1), (2) this problem has at least one continuously differentiable solution. Since we have  $x(t) = x_0^{[-1, k]}(t) + u^{[-1, k]}(t)$ ,  $t \in [k - 1, k]$ , from (18), if  $x$  is continuously differentiable at  $k - 1$  we can assume that

$$\begin{aligned} x(k - 1) &= x_0^{[-1, k]}(k - 1) + u^{[-1, k]}(k - 1) = x_0^{[-1, k]}(k - 1), \\ x'(k - 1) &= x_0'^{[-1, k]}(k - 1) + u'^{[-1, k]}(k - 1) = x_0'^{[-1, k]}(k - 1), \\ x(k) &= x_0^{[-1, k]}(k) + u^{[-1, k]}(k) = x_0^{[-1, k]}(k). \end{aligned} \quad (22)$$

Taking into account the second of the boundary conditions (2) these equalities can be rewritten as

$$\begin{aligned} f(k - 1) &= x_0^{[-1, k]}(k - 1) + u^{[-1, k]}(k - 1) = x_0^{[-1, k]}(k - 1), \\ f'(k - 1) &= x_0'^{[-1, k]}(k - 1) + u'^{[-1, k]}(k - 1) = x_0'^{[-1, k]}(k - 1), \\ f(k) &= x_0^{[-1, k]}(k) + u^{[-1, k]}(k) = x_0^{[-1, k]}(k). \end{aligned} \quad (23)$$

Note that the values on the right-hand side of (23) depend only on the the first boundary condition (2) and on the coefficients of equation (1). Thus these equalities give a set of **necessary conditions on the data of the BVP (1), (2) for the existence of at least one continuously differentiable solution**.

On the other hand, if a BVP is given which does not satisfy the conditions (23), the method described will nevertheless produce a numerical *solution*, which in this case has no relationship with the exact solution of the problem. Such solution however will not satisfy all the boundary conditions (21), but only the conditions of the reduced BVP. Thus, the set of conditions (21) gives us a criterion to verify whether a certain function, obtained by the described method, is an approximate solution to the original BVP. The next example shows us how this criterion can be applied in a specific situation.

**Example 2.1.** Consider the following MTFDE

$$x'(t) = (m - 0.5e^{-m} - 0.5e^m)x(t) + 0.5x(t - 1) + 0.5x(t + 1), \quad (24)$$

with boundary conditions  $\varphi_1(t) = e^{mt}$ ,  $t \in [-1, 0]$ ;  $f(t) = e^{mt} + e^{nt}$ ,  $t \in (k - 1, k]$ , where  $m, n \in \mathbb{R}$ ,  $m \neq 0$ ,  $m \neq n$ . This problem has no continuously differentiable solution, for any  $n \in \mathbb{R}$ . Actually, since  $\varphi_1(t) = e^{mt}$ , we know that  $x_0^{[-1, k]}(k - 1) = e^{m(k-1)}$ , while  $f(k - 1) = e^{m(k-1)} + e^{n(k-1)}$ ; obviously the first condition in (23) is not satisfied and the same can be verified for the other conditions.

Finally, note that if a continuously differentiable solution  $x$  exists and  $b(t)c(t) > 0$ ,  $\forall t \in [0, k - 1]$ , then, by Theorem 3.1, this solution is unique in the space  $W^{1,p}(0, k)$ .

### 3. Uniqueness of solution

In [14], when analysing mixed-type functional differential equations on a real interval  $[-\tau, \tau]$ ,  $\tau \in \mathbb{R}$ , Mallet-Paret and Verduyn Lunel introduced the linear operator  $\Lambda^\tau$ , defined by

$$(\Lambda^\tau w)(t) = w'(t) - \alpha(t)w(t) - \beta(t)w(t-1) - \gamma(t)w(t+1), \quad (25)$$

where  $w \in W_0^{1,p}(-\tau, \tau)$ . Here,  $W_0^{1,p}(-\tau, \tau)$  denotes the subspace of the Sobolev<sup>2</sup> space  $W^{1,p}(-\tau, \tau)$  formed by functions  $w$  defined on  $[-\tau, \tau]$  that satisfy  $w(-\tau) = w(\tau) = 0$  ( $p \geq 1$ ).

In evaluating (25) we extend  $w$  to a function defined on  $[-\tau-1, \tau+1]$  by setting  $w(t) = 0$ , if  $t \in [-\tau-1, -\tau)$  or  $t \in (\tau, \tau+1]$ . In Sec. 5 of [14], the operator  $\Lambda^\tau$  is applied to the analysis of equation (1) on a bounded interval  $[-\tau, \tau]$ . As pointed out by the authors, even in the case of constant coefficients, ‘it is surprisingly difficult to characterise the elements of the kernel of  $\Lambda^\tau$  or to determine its dimension’. This explains the difficulty in studying the uniqueness of solutions of the boundary value problem (1) subject to (2). Actually, this BVP can be reformulated (by a simple shift of the independent variable) as a BVP on an interval  $[-\tau-1, \tau+1]$ , with boundary conditions given at  $[-\tau-1, -\tau]$  and  $[\tau, \tau+1]$ . In this case, the existence of multiple solutions, belonging to  $W^{1,p}(-\tau, \tau)$ , of a given BVP of the form (1) subject to (2) is equivalent to the existence of a nontrivial kernel of  $\Lambda^\tau$ . Next we will see how the results presented in [14]<sup>3</sup> on the dimension of  $K(\Lambda^\tau)$  can be applied to the analysis of our BVP.

Let us consider first the case  $\tau = \frac{1}{2}$ . Note that this case corresponds, in our notation, to a BVP with  $k = 2$ , where the boundary conditions are given on  $[-1, 0]$  and  $[1, 2]$  and the solution is sought on  $[0, 1]$ . For this case, according to [14], we have  $K(\Lambda^\tau) = \{0\}$ , that is, the dimension of the operator’s kernel is 0. Applying this result to the boundary value problem (1) subject to (2), we conclude that this problem has at most one solution in  $W^{1,p}(0, 1)$ .

Consider now the case  $\tau = 1$ , which corresponds in our notation to the case  $k = 3$ . In this case it is shown in [14] that  $\dim K(\Lambda^\tau) \leq 1$ , which means that a nontrivial kernel (with dimension 1) can exist for some operators of the type (25). The authors provide an example of such an operator. In terms of our BVP, this means that in the case  $k = 3$  we can have a nontrivial solution in  $W^{1,p}(0, 2)$  for a problem with zero boundary conditions. In other words, there may be multiple solutions in this space for a given set of boundary conditions.

We recall a theorem from [14] that gives sufficient conditions for the non-existence of a nontrivial kernel of  $\Lambda^\tau$ .

**Theorem 3.1.** *With  $a(t)$  real-valued, assume that either  $b(t) > 0$  and  $c(t) > 0$  for almost every  $t \in \mathbb{R}$ , or  $b(t) < 0$  and  $c(t) < 0$  for almost every  $t \in \mathbb{R}$ . Then we have that  $K(\Lambda^\tau) = \{0\}$  for every  $\tau > 0$ .*

**Corollary 3.1.** *Applying this to the BVP (1) subject to (2) we conclude that if  $b$  and  $c$  satisfy the prescribed conditions, the problem under consideration has at most one solution in  $W^{1,p}(0, k-1)$ .*

**Remark 3.1.** *Of course, if there is at most one solution to a given BVP in  $W^{1,p}(0, k-1)$ , then it follows that there is at most one solution in the space of continuously differentiable functions on  $[-1, k]$ .*

### 4. Numerical methods

#### 4.1. Outline of the methods

In this section we give the outline of two numerical methods based on the ODE approach, described in the previous section.

<sup>2</sup>By  $W^{l,p}(-\tau, \tau)$ ,  $1 \leq p \leq \infty$ ,  $l \geq 1$ , we denote the Sobolev space of functions  $f$  on the set  $(-\tau, \tau)$ , such that the  $p$ -th power of the absolute value of  $f$  and its generalised derivatives (up to and including order  $l$ ) are integrable (see, for, example [3, V.8, p.379]).

<sup>3</sup>This paper is also available at [http://www.math.leidenuniv.nl/~verduyn/publications/reports/JMP\\_VL\\_proc.pdf](http://www.math.leidenuniv.nl/~verduyn/publications/reports/JMP_VL_proc.pdf)



Following these methods, we search for an approximate solution of (1) on  $[-1, 1]$  in the form

$$\tilde{x}_N(t) = x_0(t) + \sum_{j=0}^{d-1} C_j y_j(t), \quad t \in [-1, 1], \quad (26)$$

where  $x_0$  is an initial approximation of the solution ;  $\{y_j\}_{0 \leq j \leq d-1}$  is a basis in the space of functions where the correction to the initial approximation is sought;  $d$  is the dimension of this space. The algorithms for computing  $\tilde{x}_N$  consist of three steps:

1. Constructing the initial approximation;
2. Defining a set of basis functions ;
3. Computing the  $C_j$  coefficients.

#### 4.1.1. Construction of the initial approximation

As discussed in Section 2, if a solution of equation (1), constructed by the method of steps, belongs to  $C^n((l-1, l])$  (for certain  $l \geq 1, n \geq 1$ ), then it also belongs to  $C^{n-1}((l, l+1])$ . Therefore, since we want  $\tilde{x}_N$  to be at least continuous on  $[-1, k]$  (for a certain  $k \geq 2$ ), we require that  $x_0$  belongs to  $C^k((-1, l])$ . With this in mind, we define  $x_0$  on  $[-1, 1]$  in the following way:

$$x_0(t) = \begin{cases} \varphi_1(t), & t \in [-1, 0]; \\ P_{2k}(t) = a_0 + a_1 t + \dots + a_{2k} t^{2k}, & t \in [0, 1]. \end{cases} \quad (27)$$

To ensure that  $x_0$  is sufficiently smooth (see condition (8)),  $P_{2k}$  must satisfy the following conditions at  $t = 0$ :

$$P_{2k}(0) = \varphi_1(0); \quad P_{2k}^{(j)}(0) = \varphi_1^{(j)}(0), \quad j = 1, \dots, k. \quad (28)$$

On the other hand, in order to satisfy the conditions at  $t = 1$ , taking into account formula (9), the following equalities are obtained:

$$\begin{aligned} P_{2k}(1) &= x(1) = a(0)\varphi_1'(0) + b(0)\varphi_1(-1) + c(0)\varphi_1(0); \\ P_{2k}^{(j)}(1) &= \frac{d^j}{dt^j} (a(t)\varphi_1'(t) + b(t)\varphi_1(t-1) + c(t)\varphi_1(t))|_{t=0}, \quad j = 1, \dots, k-1, \end{aligned} \quad (29)$$

where  $a, b, c$  are defined by (6). Conditions (28), (29) define a linear system of  $2k+1$  equations with  $2k+1$  unknowns. It is possible to show that this system has a nonsingular matrix, for any  $k \geq 2$ .

Further,  $x_0$  is extended from  $[-1, 1]$  to  $[-1, k]$  using the recurrence formulae (10). Let us denote by  $x_0^{[-1, k]}$  this extension.

#### 4.1.2. Definition of a basis

With the purpose of computing a correction to the initial approximation on  $[0, k]$ , we first consider this correction on  $[0, 1]$ . Let us define a grid of stepsize  $h$  on this interval. Let  $h = 1/N$  (where  $N \in \mathbb{N}$ ,  $N \geq k+1$ ) and  $t_i = ih$ ,  $i = 0, \dots, N$ . The correction  $\tilde{x}_N(t) - x_0(t)$  on  $[0, 1]$  will be sought as a  $k$ -th degree spline,  $S_k(t)$ , defined on this grid, which satisfies  $S_k(0) = S_k(1) = 0$ . As usual, we will use as basis functions  $y_j(t)$  the so-called  $B$ -splines of degree  $k$ . Following the usual definition (for details see [15, p.298]), we have

$$y_j(t) = \frac{1}{h^k} \Delta^{k+1}(t - t_j)_+^k, \quad j = 0, \dots, N - k - 1, \quad (30)$$

where  $\Delta^k$  represents an order  $k$  forward difference (with respect to  $t_j$ ) and

$$(t - t_j)_+ = \begin{cases} 0, & \text{if } t < t_j; \\ (t - t_j), & \text{if } t \geq t_j. \end{cases}$$

From the definition it follows that the basis functions have the following properties:

- $y_j \in C^{k-1}[0, 1]$ ;
- $y_j(t)$  is different from zero only in  $(t_j, t_{j+k+1})$ ;
- $y_j(t)$  is a polynomial of degree  $k$  on each interval  $[t_i, t_{i+1}]$ ,  $i = 0, \dots, N-1$ .

Note that we have  $N-k$  functions  $y_j$  with these properties; therefore, we set  $d = N-k$ .

The case of  $k=3$  (cubic B-splines) will be analysed in more detail in Sec. 5.

Next the basis functions are extended to the interval  $[0, k]$  using the formulae (10), where  $\varphi_1$  is replaced by 0 and  $\phi_2$  is replaced by  $y_j$ . Let us denote the extended basis functions by  $y_j^{[0,k]}$ . Each time we extend the basis function to the next interval, the degree of smoothness of the splines reduces by one (though the polynomial degree remains the same). Therefore, on the interval  $[k-1, k]$ , the basis functions are continuous but not continuously differentiable. On the whole interval  $[0, k]$ , the approximate solution is given by

$$\tilde{x}_N^{[0,k]}(t) = x_0^{[-1,k]}(t) + \sum_{j=0}^{N-k-1} C_j y_j^{[0,k]}(t), \quad t \in [0, k]. \quad (31)$$

#### 4.1.3. Computation of the coefficients

Finally, we compute the coefficients  $C_j$ ,  $j = 0, \dots, N-k-1$  of the expansion (31) from the condition that  $\tilde{x}_N$  approximates  $f$  on the interval  $(k-1, k]$ . Two alternative methods were used for this purpose.

- Collocation Method (for which we give an error analysis). In this case, the coefficients are obtained from the condition

$$\begin{aligned} \tilde{x}_N^{[0,k]}(t_{(k-1)N+i}) &= x_0^{[-1,k]}(t_{(k-1)N+i}) + \sum_{j=0}^{N-k-1} C_j y_j^{[0,k]}(t_{(k-1)N+i}) \\ &= f(t_{(k-1)N+i}), \quad i = i_{min}, \dots, i_{max}, \end{aligned} \quad (32)$$

where  $i_{min} = (k+1)/2$ , if  $k$  is odd;  $i_{min} = k/2$ , if  $k$  is even;  $i_{max} = N - (k+1)/2$ , if  $k$  is odd;  $i_{max} = N - k/2 - 1$ , if  $k$  is even.

Equations (32) form a linear system with a  $(N-k) \times (N-k)$  band matrix. This system can be solved by standard methods.

**Remark.** Note that the linear system (32) can be also written in the form

$$\sum_{j=0}^{N-k-1} C_j y_j^{[0,k]}(t_{(k-1)N+i}) = f(t_{(k-1)N+i}) - x_0^{[-1,k]}(t_{(k-1)N+i}), \quad i = i_{min}, \dots, i_{max}. \quad (33)$$

Taking into account that  $y_j^{[k-1,k]}(t) = L_{k-1} y_j(t-k+1)$ ,  $j = 0, \dots, N-k-1$ ,  $t \in [k-1, k]$  where  $L_{k-1}$  is the linear differential operator defined by (17), we see that the system (33) is the same system of linear equations that we obtain if we apply the collocation method to the numerical solution of the ODE (20). Therefore if we denote

$$u_N(t) = \sum_{j=0}^{N-k-1} C_j y_j(t), \quad t \in [0, 1], \quad (34)$$

we conclude that  $u_h$  is the approximate solution of the reduced boundary value problem (see Section 3) using the collocation method with a basis of B-splines and stepsize  $h$ .

We shall use this approach when analysing the error of the method.

- Least Squares Method (which we include in the examples for comparison). In this case, the coefficients  $C_j$  are obtained from the condition that the following integral is minimised:

$$\int_{k-1}^k \left( f(t) - x_0^{[-1,k]}(t) - \sum_{j=0}^{N-k-1} C_j y_j^{[0,k]}(t) \right)^2 dt.$$

Given the form of the basis functions this method leads us to the solution of a system of  $N - k$  linear equations with a band matrix. Once again, we can remark that the resulting system of linear equations is the same system that we would obtain if we applied the least squares method to the solution of the equation (20).

#### 4.2. Error analysis for the collocation method

Next we analyse the error of the approximate solution  $\tilde{x}_N(t)$ , obtained by the collocation method described in the previous section.

First we restrict the error analysis to the case  $t \in [0, 1]$ . According to (26), in the interval  $[-1, 1]$  this solution can be written in the form

$$\tilde{x}_N(t) = x_0(t) + u_N(t) = x_0(t) + \sum_{j=0}^{N-k-1} C_j y_j(t), \quad t \in [-1, 1]. \quad (35)$$

$u_N$  is an approximation to the solution of the reduced BVP (20), obtained by the collocation method, using splines of the class  $L(\pi_n, k, k-1)$  (using the notation of [16]). We assume that the coefficient functions of the BVP under consideration are at least continuous and that this BVP, with  $k-1$  boundary conditions, has a unique solution. Then the conditions of Theorem 2 of [16] are satisfied and, according to the Corollary of this theorem, we have the following error bound:

$$\|u_N - u\|_\infty = \max_{t \in [0, 1]} |u_N(t) - u(t)| \leq B_0 h^2, \quad t \in [0, 1], \quad (36)$$

where  $B_0$  is a constant that does not depend on  $h$ .

Let us now analyse the error of  $u_N^{[m, m+1]}$  (the approximation of  $u^{[m, m+1]}$ , obtained by extending  $u_N$  to the interval  $[m, m+1]$  for  $m \leq k-1$ ,  $m \in N$ ). On the interval  $[m, m+1]$ ,  $u^{[m, m+1]}(t)$  and  $u_N^{[m, m+1]}(t)$  are given by

$$\begin{aligned} u^{[m, m+1]}(t) &= L_m u(t-m) = \sum_{l=0}^m c_{l,m} u^{(l)}(t-m), \\ u_N^{[m, m+1]}(t) &= L_m u_N(t-m) = \sum_{l=0}^m c_{l,m} u_N^{(l)}(t-m), \quad t \in [m, m+1]. \end{aligned} \quad (37)$$

where  $L_m$  is the linear differential operator defined by (17). From (37), we conclude that

$$\begin{aligned} \|u_N^{[m, m+1]} - u^{[m, m+1]}\|_\infty &= \max_{t \in [m, m+1]} |u_N^{[m, m+1]}(t) - u^{[m, m+1]}(t)| = \\ &= \|L_m u_N - L_m u\|_\infty = \|\sum_{l=0}^m c_{l,m}(t)(u_N^{(l)} - u^{(l)})\|_\infty \leq \\ &\sum_{l=0}^m \|c_{l,m}\|_\infty \|u_N^{(l)} - u^{(l)}\|_\infty \leq (m+1) \max_{l=0, \dots, m} \|c_{l,m}\|_\infty \max_{l=0, \dots, m} \|u_N^{(l)} - u^{(l)}\|_\infty. \end{aligned} \quad (38)$$

Recall that by formula (2.16) of [16], the error bound (36) also holds for the derivatives of the solution:

$$\|u_N^{(l)}(t) - u^{(l)}(t)\|_\infty \leq B_l h^2, \quad t \in [0, 1], \quad l = 0, 1, \dots, m, \quad (39)$$

where  $B_l$  are constants that do not depend on  $h$ . Finally, by substituting (39) into (38), we obtain

$$\|u_N^{[m, m+1]} - u^{[m, m+1]}\|_\infty \leq C h^2, \quad (40)$$

where  $C = (m+1) \max_{l=0, \dots, m} \|c_{l,m}(t)\|_\infty \max_{l=0, \dots, m} B_l$ .

Finally, we remark that the approximate solution in any interval  $[m, m+1]$  is given by

$$\tilde{x}_N^{[m, m+1]}(t) = x_0^{[m, m+1]}(t) + u_N^{[m, m+1]}(t), \quad m = 1, \dots, k-1. \quad (41)$$

Since  $x_0^{[m, m+1]}$  can be computed exactly the error of  $x_N^{[m, m+1]}$  on each interval  $[m, m+1]$  satisfies

$$\|\tilde{x}_N^{[m, m+1]} - x\|_\infty \leq C h^2. \quad (42)$$

The above error analysis for the collocation method can be summarised in the following theorem.

**Theorem 4.1.** *Assuming that the problem (1),(4) is solvable (in the sense specified in Section 2), the collocation method described in subsection 4.1 has second order convergence, that is, the approximate solution  $\tilde{x}_N$  satisfies the error bound (42).*

#### 4.3. Two equivalent algorithms

The collocation method considered here can be applied in two different forms, which differ only in the order of the computations.

##### Algorithm 1.

1. Compute the coefficients of the linear ODE (20).
2. Solve the reduced BVP for equation (20) by the collocation method (using the appropriate MATLAB package)
3. Compute the extension of the approximate solution  $u_N$  to the interval  $[0, k]$  and add to  $x_0^{[-1, k]}$  to obtain  $\tilde{x}_N^{[0, k]}$ .

##### Algorithm 2.

1. Compute the extensions  $y_j^{[0, k]}$  of the basis functions and the extension  $x_0^{[-1, k]}$  of the initial approximation.
2. Solve the linear system (33).

In the next section we will present numerical results obtained using these two algorithms. These results have the same accuracy (up to the computational error). We will also display some results obtained from the least squares method, which has a higher estimated convergence order.

Methods presented here and in [4], [5], [19] and [20] are concerned with MTFDEs with boundary conditions given on the intervals  $[-1, 0]$  and  $(k - 1, k]$ ,  $k \in \mathbb{N}$ , and with the size of both delay and advance equal to unity. In other words, the interval over which a solution is sought,  $(0, k - 1]$ , is an integer multiple of the delay in the equation. Our approach can be adapted to the case when the delay and advance are equal to a constant  $d$ , and the solution to the equation is sought over an interval  $(0, \ell - d]$ , with boundary conditions provided on  $[-d, 0]$  and  $(\ell - d, \ell]$ , as long as  $\ell$  and  $d$  are commensurate. In this case, the step length of the numerical method  $h$  can be chosen such that, for a suitable integer  $g$ ,  $d = n_1 hg$ ,  $\ell = n_2 hg$ , where  $n_1$  and  $n_2$  are relatively prime. The method of steps can be used to propagate the solution from  $((j - 1)gh, jgh]$  to  $((j - 1)gh + d, jgh + d]$ , for  $j = 1, 2, \dots, \ell/d$ . However, a larger number of constraints is needed to ensure continuity of the function and its derivatives at the points  $t = jgh$ ,  $j = 0, 1, 2, \dots, n_2$ .

**Example 4.1.** Consider the equation

$$x'(t) = \alpha(t)x(t) + \beta(t)x(t - 0.4) + \gamma(t)x(t + 0.4),$$

and the boundary conditions

$$x(t) = \begin{cases} \varphi_1(t), & \text{if } t \in [-0.4, 0], \\ f(t), & \text{if } t \in (2.6, 3], \end{cases} \quad (43)$$

where  $\varphi_1$  and  $f$ , as usual, are known functions. In this case, using the above notation, we have  $d = 0.4$ ,  $\ell = 3$  and thus  $\ell/d = 3/.4 = 15/2$ . We set  $hg = 0.2$ , which means that we can choose as stepsize any divisor of 0.2. In this case, we have  $n_1 = d/(gh) = 2$ ,  $n_2 = \ell/(gh) = 15$  (2 and 15 are relatively prime).

To apply the method of steps to this problem, we proceed as follows. We define a function  $\varphi_2(t)$  on  $(0, 0.2]$ ; using  $\varphi_1$  and  $\varphi_2$ , we can propagate the solution to all the intervals of the form  $(j * 0.4, j * 0.4 + 0.2]$ ,  $j = 1, \dots, 7$ . This includes  $(2.8, 3]$ , where  $f$  is defined. Next we define a function  $\varphi_3(t)$  on  $(0.2, 0.4]$ ; using  $\varphi_1$  and  $\varphi_3$ , we can propagate the solution to all the intervals of the form  $(j * 0.4 + 0.2, (j + 1) * 0.4]$ ,  $j = 1, \dots, 6$ . This includes  $(2.6, 2.8]$ , where  $f$  is also defined. That is, we reformulate the original BVP as two initial value problems, which can be solved by the method of steps. Then, to solve the original BVP, we must find  $\varphi_2$  and  $\varphi_3$  such that the second part of boundary condition (43) is satisfied.

$\mathbf{k} = 5$	interval $[0, 1]$		interval $[1, 2]$		interval $[2, 3]$		interval $[3, 4]$	
$N$	$\epsilon$	$p$	$\epsilon$	$p$	$\epsilon$	$p$	$\epsilon$	$p$
8	$3.9626e-8$		$4.1222e-7$		$6.8709e-6$		$1.4375e-4$	
16	$1.1002e-8$	1.8487	$1.0570e-7$	1.9634	$1.6164e-6$	2.0877	$2.8220e-5$	2.3488
32	$2.4341e-9$	2.1763	$2.3990e-8$	2.1395	$3.5366e-7$	2.1923	$6.3817e-6$	2.1447
64	$5.4937e-10$	2.1475	$5.4222e-9$	2.1455	$8.0624e-8$	2.1331	$1.4596e-6$	2.1284
128	$1.2893e-12$	2.0912	$1.2776e-9$	2.0854	$1.9112e-8$	2.0767	$3.4746e-7$	2.0707

Table 1: Numerical results for Example 5.1 by the collocation method (algorithm 1) with  $m = 2$  and  $k = 5$ .  $\epsilon = \|x - \tilde{x}_N\|_\infty$  on each interval. Part 1: solution

$\mathbf{k} = 5$	interval $[0, 1]$		interval $[1, 2]$		interval $[2, 3]$		interval $[3, 4]$	
$N$	$\epsilon$	$p$	$\epsilon$	$p$	$\epsilon$	$p$	$\epsilon$	$p$
8	$1.8990e-7$		$3.2875e-6$		$7.3879e-5$		$2.4814e-3$	
16	$4.6366e-8$	2.0341	$7.9830e-7$	2.0420	$1.4721e-5$	2.3273	$9.1067e-4$	1.4462
32	$1.0027e-8$	2.2092	$1.7881e-7$	2.1585	$3.0748e-6$	2.2593	$2.5336e-4$	1.8458
64	$2.2789e-9$	2.1375	$4.0841e-8$	2.1303	$7.0036e-7$	2.1343	$6.5608e-5$	1.9492
128	$5.3685e-10$	2.0857	$9.6704e-9$	2.0784	$1.6663e-7$	2.0715	$1.6622e-5$	1.9808

Table 2: Numerical results for Example 5.1 by the collocation method (algorithm 1) with  $m = 2$  and  $k = 5$ .  $\epsilon = \|x - \tilde{x}_N\|_\infty$  on each interval. Part 2: derivative

## 5. Numerical results

Next we present two examples of BVP for MTFDEs (Examples 5.1 and 5.2) which were used to test the described numerical algorithms. The error norm on each interval  $[l, l+1]$  is the discrete analog of the maximum norm:

$$\|\tilde{x}_N^{[l,l+1]} - x\|_\infty = \max_{i=0,\dots,N} |\tilde{x}_N^{[l,l+1]}(t_{lN+i}) - x(t_{lN+i})|.$$

The estimate of the convergence order is obtained by evaluating the quantity

$$p = \log_2 \frac{\|x - \tilde{x}_N\|_\infty}{\|x - \tilde{x}_{2N}\|_\infty}, \quad (44)$$

where  $\tilde{x}_N$  and  $\tilde{x}_{2N}$  are the approximate solutions obtained using two different grids, with stepsize  $h$  and  $h/2$ , respectively. If we assume that, for some  $p_0 > 0$ , there exists a constant  $C$  such that the error norm satisfies

$$\|x - \tilde{x}_N\|_\infty = C(1/N)^{p_0}(1 + o(1)), \text{ as } N \rightarrow \infty,$$

then we can conclude that, for sufficiently high values of  $N$ , the quantity defined by (44) gives a good approximation to the exact convergence order  $p_0$ .

**Example 5.1.** *The following example of an autonomous equation was first considered in [4]:*

$$x'(t) = (m - 0.5e^{-m} - 0.5e^m)x(t) + 0.5x(t-1) + 0.5x(t+1), \quad (45)$$

with boundary conditions  $\varphi_1(t) = e^{mt}, t \in [-1, 0]$ ;  $f(t) = e^{mt}, t \in (k-1, k]$ , with  $m \in \mathbb{R}$ ,  $m \neq 0$ . The exact solution is  $x(t) = e^{mt}$ .

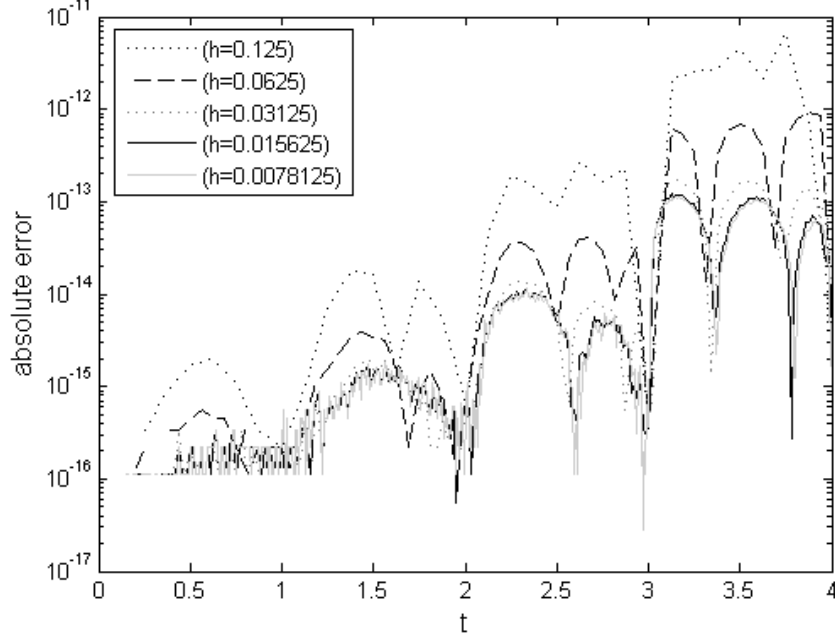


Figure 1: Absolute error of numerical solution for Example 5.1 by the least squares method with  $k = 5$  and  $m = -0.5$ .

In Tables 1 and 2, the error norm and the estimated convergence order of the numerical solution and its first derivative, respectively, obtained by the first algorithm, for Example 5.1 with  $m = 2$ , are displayed. Error norms were computed separately on each interval  $[l, l + 1]$  and globally on  $[0, k - 1]$ . The numerical solution and its first derivative present error norms on  $[0, 4]$  equal to the error norm on  $[3, 4]$ . As expected, the convergence order when approximating the first derivative is the same as when approximating the solution.

The figures 1 and 2 describe the behaviour of the error when  $k = 5$ ,  $m = -0.5$  and  $m = 2$  respectively.

**Example 5.2.** Concerning the non-autonomous case, the following mixed-type equation was considered

$$x'(t) = m x(t) - e^{-m(t-2)}x(t-1) + e^{m(t-1)}x(t+1), \quad (46)$$

where  $\varphi_1(t) = e^{mt}$ ,  $t \in [-1, 0]$ ;  $f(t) = e^{mt}$ ,  $t \in [k-1, k]$ , with  $m \in \mathbb{R}$ ,  $m \neq 0$ . The exact solution is  $x(t) = e^{mt}$ .

The results of the numerical experiments with this example can be compared with those of example 5.1, for each value of  $m$ . The numerical results obtained by the collocation method (algorithm 1) are displayed in table 3. We observe that the absolute error of the numerical solution is of the same order for both methods, but it is always smaller for the autonomous case. The estimated order of convergence is close to  $p = 2$ , in both cases, in agreement with the theoretical results.

Table 4 contains numerical results obtained by the least squares method. The absolute error of the numerical solution is always smaller for example 5.1 than for example 5.2, though they are very close. For both examples, the estimated order of convergence is close to  $p = 3$ . An analysis of the error of the least squares method will be carried out in a future article.

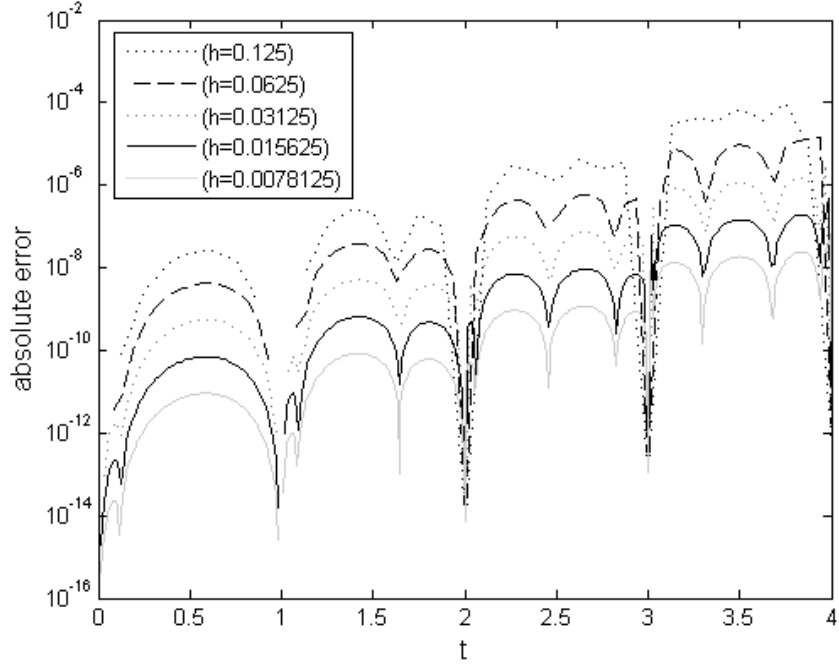


Figure 2: Absolute error of numerical solution for Example 5.1 by the least squares method with  $k = 5$  and  $m = 2$ .

$\mathbf{k} = \mathbf{3}$	Example 5.1		Example 5.2		Example 5.1		Example 5.2	
	interval $[0, 1]$				interval $[1, 2]$			
$N$	$\epsilon$	$p$	$\epsilon$	$p$	$\epsilon$	$p$	$\epsilon$	$p$
8	$1.0759e-4$		$1.5806e-4$		$9.0067e-4$		$3.9686e-3$	
16	$2.4863e-5$	2.1135	$4.2546e-5$	1.8934	$2.0241e-4$	2.1537	$9.9803e-4$	1.9915
32	$5.8093e-6$	2.0976	$1.0890e-5$	1.9660	$4.7534e-5$	2.0903	$2.5228e-4$	1.9841
64	$1.3968e-6$	2.0562	$2.7389e-6$	1.9913	$1.1440e-5$	2.0549	$6.3243e-5$	1.9960
128	$3.4158e-7$	2.0319	$6.8650e-7$	1.9963	$2.8040e-6$	2.0285	$1.5823e-5$	1.9988

Table 3: Numerical results for Examples 5.1 and 5.2 with  $m = 2$ , by the collocation method.  $\epsilon = \|x - \tilde{x}_N\|_\infty$  on each interval.

$\mathbf{k} = \mathbf{3}$	Example 5.1		Example 5.2		Example 5.1		Example 5.2	
	interval $[0, 1]$				interval $[1, 2]$			
$N$	$\epsilon$	$p$	$\epsilon$	$p$	$\epsilon$	$p$	$\epsilon$	$p$
8	$3.8330e-5$		$4.3503e-5$		$4.8313e-4$		$1.7584e-3$	
16	$1.2449e-5$	1.6224	$8.8879e-6$	2.2912	$8.4657e-5$	2.5127	$3.2108e-4$	2.4532
32	$6.6044e-7$	4.2365	$1.3607e-6$	2.7075	$1.2230e-5$	2.7912	$4.6233e-5$	2.7959
64	$8.3437e-8$	2.9847	$1.8625e-7$	2.8691	$1.6350e-6$	2.9031	$6.1267e-6$	2.9158
128	$1.0486e-8$	2.9923	$2.4300e-8$	2.9382	$2.1110e-7$	2.9533	$7.8601e-7$	2.9625

Table 4: Numerical results for Examples 5.1 and 5.2 with  $m = 2$ , by the least squares method.  $\epsilon = \|x - \tilde{x}_N\|_\infty$  on each interval.

## 6. Conclusions and future work

In the present paper, we have presented a new approach for the analysis and approximation of a boundary value problem for a linear MTFDE. The question of existence and uniqueness of solution has been addressed and some conditions for the existence of a unique continuously differentiable solution were established. It was shown that the original problem (1)-(2) can be reduced to a certain boundary value problem for an ordinary differential equation, whose order depends on the length of the considered interval. Based on this approach, two numerical algorithms were proposed for the solution of the problem. The first numerical algorithm, based on the collocation method, was shown to have estimated order of convergence two. Numerical results presented are in agreement with the theoretical analysis.

In the future, we intend to carry out a detailed numerical analysis of the least squares method. Additionally, we plan to improve the convergence of both methods, by using special grid points (for example, Gaussian points).

We are also planning to explore in greater depth the stability of the methods, to enable us to solve the problem numerically on longer intervals. We also intend to develop numerical methods, based on our approach, for nonlinear problems.

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